# Response of degree-correlated scale-free networks to stimuli

Sheng-Jun Wang,[1] An-Cai Wu,[1] Zhi-Xi Wu,[1] Xin-Jian Xu,[2] and Ying-Hai Wang[1,*]
[1]*Institute of Theoretical Physics, Lanzhou University, Lanzhou Gansu 730000, China*
[2]*Departamento de Física da Universidade de Aveiro, 3810-193 Aveiro, Portugal*

(Received 27 October 2006; revised manuscript received 3 February 2007; published 23 April 2007)

The response of degree-correlated scale-free attractor networks to stimuli is studied. We show that degree-correlated scale-free networks are robust to random stimuli as well as the uncorrelated scale-free networks, while assortative (disassortative) scale-free networks are more (less) sensitive to directed stimuli than uncorrelated networks. We find that the degree correlation of scale-free networks makes the dynamics of attractor systems different from uncorrelated ones. The dynamics of correlated scale-free attractor networks results in the effects of degree correlation on the response to stimuli.

DOI: 10.1103/PhysRevE.75.046113 PACS number(s): 89.75.Hc, 87.18.Sn, 05.50.+q, 05.40.−a

Many complex systems have the ability to react to low levels of special stimuli, whereas they can maintain their state when exposed to high levels of other irrelevant stimuli [1]. If we take the units of response as nodes and the interactions between responding units as edges, the structure of some these systems can be described as complex networks. In neural networks or social networks, for example, the nodes are individual neurons or persons. It is an interesting problem how one system has both sensitivity to the right stimuli and robustness in the face of the wrong one. And the problem is also important for designing large artificial complex systems. The source of the ability of networked complex systems to incorporate the two complementary attributes has been investigated using network models. It was shown that the power-law shape degree distributions of networks give rise to the sensitivity and robustness in a system [1].

The topology of real networks is also characterized by degree correlation [2–4]. In a network with degree correlation, there exist certain relationships between network nodes. The degree correlations are often named, respectively, "assortative mixing"—i.e., a preference for high-degree nodes to attach to other high-degree nodes—and "disassortative mixing"—i.e., high-degree nodes attach to low-degree ones [4]. It has been pointed out that the existence of degree correlations among nodes is an important property of real networks [5–14]. The percolation [4] and disease spreading [15] on correlated networks have been studied. And more effects of degree correlation on network structure and function have attracted attention [16–18]. Therefore, an extension of previous results for uncorrelated network model about responding to stimuli is necessary.

In this paper, we study the response of degree-correlated scale-free networks to stimuli following the work contributed by Bar-Yam and Epstein [1]. Numerical investigation reveals that the dynamical process of the evolution of attractor systems on correlated scale-free networks is different from uncorrelated networked systems. The special dynamics of correlated attractor systems results in a different responding behavior from uncorrelated systems. The degree-correlated scale-free network is robust in the face of wrong stimuli as uncorrelated networks. In assortative networks, the sensitivity to right stimuli is enhanced, while in the disassortative networks the sensitivity to right stimuli is weaker than uncorrelated networks. And the relation between the sensitivity to stimuli and the degree of correlation is not monotonic.

We consider the method for modeling the response of complex systems proposed in [1]. We use a model of attractor networks [19,20], where the node states $s_i = \pm 1$, $i \in \{1, \dots, N\}$ are binary. The state of the system is the set of node states $\{s_i\}$. The dynamical equation of the attractor system is

$$s_i(t+1) = \mathrm{sgn}\left(\sum_{j=1}^{N} J_{ij} s_j(t)\right), \tag{1}$$

with symmetric influence matrix $J_{ij}$. Using the Hebbian imprinting rule

$$J_{ij} = \sum_{\alpha} c_{ij} s_i^{\alpha} s_j^{\alpha}, \tag{2}$$

we can set the states $\{s_i^{\alpha}\}_{\alpha=1,\dots,n}$ as the stable states of the network dynamics (attractor). $c_{ij}$ is the entry of the symmetric adjacent matrix which is equal to 1 when node $i$ connects to node $j$ and zero otherwise. An attractor is stable to perturbation and thus can represent a functional state of systems. In simulations, we randomly choose two functional states of the system, $\{s_i^{\alpha}\}_{\alpha=1,2}$, and the influence is $J_{ij} = \Sigma_{\alpha=1}^{2} c_{ij} s_i^{\alpha} s_j^{\alpha}$. External stimuli are modeled by changing the signs of a specified set of nodes. When the states of some nodes are flipped, the system either evolves back to its initial state or switches to other stable system states. The response of networked systems is described as a process of switching between attractors. The size of the basin of attraction, the number of nodes whose states can be changed before the dynamics of the network fails to bring the system back to its original state, indicates the degree of stability of the system. We calculate the size of the basin of attraction in different cases of stimuli to reveal the sensitivity and robustness of the network model.

Generally, degree-correlated networks can be generated from uncorrelated ones by means of reshuffling method proposed in [5]. Starting from a given network, at each step two edges of the network are chosen at random. The four nodes

*Electronic address: yhwang@lzu.edu.cn

 

attached to the two edges are ordered with respect to their degrees. Then with probability $p$ the edges are rewired in such a way that one edge connects the two nodes with the smaller degrees and the other connects the two nodes with the larger degrees; otherwise, the edges are randomly rewired. In the case when one or both of these new edges already existed in the network, the step is discarded and a pair of other edges is selected. A repeated application of the rewiring step leads to an assortative networks. For producing disassortative networks, we change the way for building new edges used in the above reshuffling method into that the node of the largest degree connects to the nodes of the smallest degree and two other node are connected. It is worth noting that the algorithm does not change the degree distribution in the given network [5].

Before investigating the effect of the degree correlation on the response, we review the results on uncorrelated attractor networks [1], where the system was characterized by the scale-free networks which have the power-law shape degree distribution $P(k) \sim k^{-\gamma}$. The sizes of the basin of attraction for two kinds of stimuli—namely, the random stimuli (randomly chosen nodes are flipped) and the directed stimuli (means flipping sequentially the nodes of greatest degree)—were studied on scale-free attractor network systems. The relation between the size of the basin of attraction for random stimuli $b_r$ and directed stimuli $b_m$, which are all normalized by network size $N$, is derived:

$$b_m = b_r^{(\gamma-1)/(\gamma-2)}. \tag{3}$$

The derivation was based on an assumption that the response of attractor networks occurs if the sum of edges coming from stimulated nodes exceeds a threshold which is the same for both random and directed stimuli. For Barabási-Albert (BA) scale-free networks [21], the distribution exponent $\gamma=3$ and thus $b_m = b_r^2$. So the scale-free networks are robust to random stimuli and sensitive to directed stimuli.

Let us first calculate the average size of the basin of attraction for random stimuli $b_r$ and directed stimuli $b_m$ on degree-correlated BA networks. According to [1], we use the network size $N=1000$ and average degree $\langle k \rangle = 20$ in all simulations. Figure 1 shows the average size of attractor basin versus the degree of correlation which is quantified by the Pearson correlation coefficient $r$ [4]. To compare with the uncorrelated case, in Fig. 1 we also plot the predicted size of the attractor basin for directed stimuli $b'_m$ which is calculated using the size of the attractor basin for random stimuli $b_r$ following Eq. (3). Restricted by the reshuffling method, we cannot generate networks with strong degree correlation $|r| \to 1$ [5]. In simulations, the region of the Pearson correlation coefficient $r$ is about from $-0.3$ to 0.3. Although the region is small, it nearly covers all the values of the Pearson correlation coefficient $r$ of realistic complex networks shown in[4]. Therefore, we are interested in systems with the Pearson correlation coefficient belonging to the region about from $-0.3$ to 0.3.

In Fig. 1 we can see the effects of the degree correlation of scale-free networks on the size of the basion of attraction. Comparing the size of attractor basin $b'_m$ predicted using Eq. (3) (the curve with triangles) with the size obtained by computer simulations (the curve with squares), one can see that the relation between the size of attractor basin for random stimuli $b_r$ and directed stimuli $b_m$ derived in uncorrelated case is not satisfied in correlated scale-free networks. When $r \approx 0$ the numerical result of the attractor basin for directed stimuli $b_m$ is identical with the prediction of uncorrelated networks $b'_m$ [22]. For the assortative case $r > 0$, the basin of attraction for directed stimuli is less than the value of uncorrelated network. This means that the assortative scale-free network is more sensitive to directed stimuli than uncorrelated scale-free networks. For the disassortative case, the size of attractor basin undergoes a nonmonotonic process with the variance of Pearson correlation coefficient. The sensitivity of disassortative scale-free networks is weaker than uncorrelated systems. The size of the basin of attraction for random stimuli $b_r$ decreases monotonically with the increase of $r$. And the slope is small. The robustness of scale-free networks to random stimuli is retained when these networks are degree correlated.

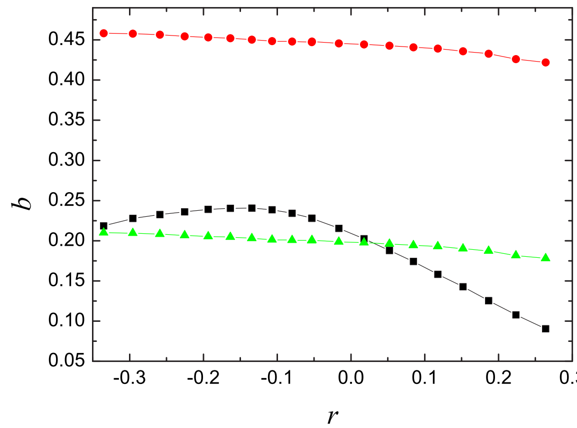


FIG. 1. (Color online) The size of attractor basin of scale-free networks as a function of Pearson correlation coefficient $r$ in the case of directed (square) and random stimuli (circle). All networks have the same network size $N=1000$ and average degree $\langle k \rangle = 20$. Each curve is an average of 1000 realizations. The predicted curve of $b'_m$ calculated using Eq. (3) is shown as the curve with triangles.

To understand the underlying mechanism of the effect of degree correlation on response, we analyze the dynamics of attractor networks. We assume that there are $n$ functional states in an attractor system. Substitute of Eq. (2) into Eq. (1) gives

$$s_i(t+1) = \mathrm{sgn}\left( \sum_{j=1}^{N} \sum_{\alpha=1}^{n} c_{ij} s_i^\alpha s_j^\alpha s_j(t) \right) = \mathrm{sgn}\left( \sum_{\alpha=1}^{n} s_i^\alpha \sum_{j \in G_i} s_j^\alpha s_j(t) \right), \tag{4}$$

where $G_i$ is the set of nodes adjacent to node $i$ (the neighbors of node $i$). We use the functional state $\{s_i^1\}$ as the original system state, and the stimulated system state is denoted as $\{s_i^\beta\}$. Thus the first step of the evolution is like

$$s_i(1) = \mathrm{sgn}\left( s_i^1 \sum_{j \in G_i} s_j^1 s_j^\beta + \sum_{\alpha=2}^{n} s_i^\alpha \sum_{j \in G_i} s_j^\alpha s_j^\beta \right). \tag{5}$$

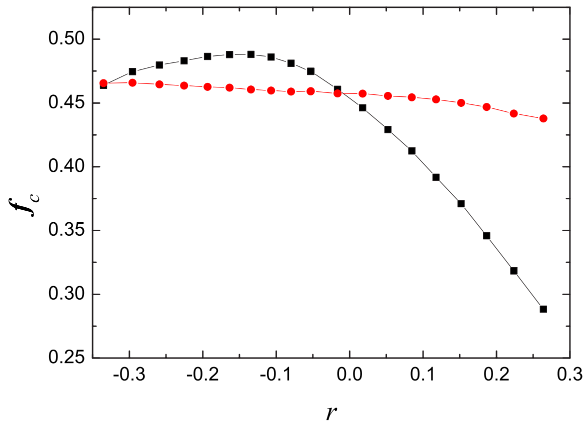


FIG. 2. (Color online) The critical value of the number of edges attached to flipped nodes as a function of Pearson correlation coefficient in the case of directed (squares) and random (circles) stimuli. Each curve is an average of 1000 realizations.

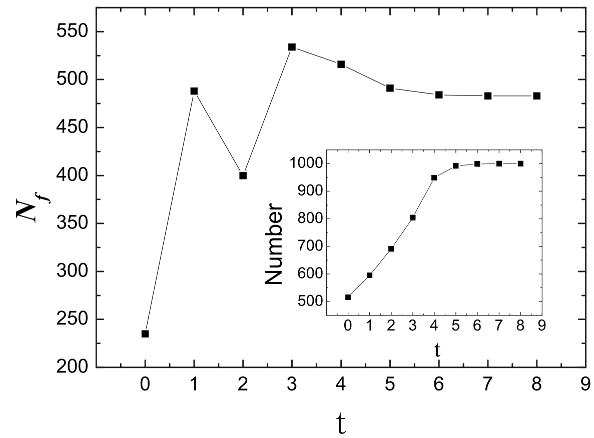


FIG. 3. The number of flipped nodes in the process of the evolution of the uncorrelated system. Inset: the number of nodes whose state $s_i(t)$ is the same as $s_i^2$.

The functional states $\{s_i^\alpha\}_{\alpha=2,\ldots,n}$ are uncorrelated with the stimulated state $\{s_i^\beta\}$, since the functional states are chosen at random. Thus the second term in the brackets on the right-hand side of Eq. (5) is approximately equal to 0, and this term can be taken as noise [19]. For an arbitrary node $i$, if much fewer than half the nodes in $G_i$ are flipped by the stimulus, then $s_i(1)=s_i^1$; if much more than half the nodes in $G_i$ are flipped, $s_i(1)=-s_i^1$. In general, the fraction of flipped nodes in $G_i$ increases as stimuli are enhanced. Because of the influence of noise, when the fraction of flipped nodes in $G_i$ is near but less than 0.5, the node $i$ choose a state $s_i^1$ or $-s_i^1$ at random.

In the case of uncorrelated networks, for both random and directed stimuli, the fraction of flipped nodes in neighbors of each node is equal to the fraction $f$ of edges coming from flipped nodes in a network. This property determines a critical condition for uncorrelated systems responding to stimuli: near half edges in a network come from the stimulated nodes. We obtained the critical value of $f$ on the system with two functional states by numerical simulation, which is $f_c = 0.46$ for both random and directed stimuli. When stimuli are large enough to satisfy the critical condition, all nodes in uncorrelated networks choose their states at random with the help of noise term. Then, the system state $\{s_i(1)\}$ becomes a random state and evolves to one of attractors randomly. The analysis of the above property gives insight into the dynamics of uncorrelated networks, that the uncorrelated networks respond to both kinds of stimuli as a whole.

Figure 2 shows numerical results of the critical fraction of edges attached to stimulated nodes versus the Pearson correlation coefficient of reshuffling scale-free networks. When networks are degree correlated, the difference between the critical fraction $f_c$ for random stimuli and directed stimuli is remarkable. The result shows that the mentioned assumption used for deriving Eq. (3) in [1] is not appropriate for degree-correlated scale-free networks. In Fig. 2, one can note that the critical fraction $f_c$ for random stimuli varies slightly. Under random stimuli, for correlated scale-free networks, the fraction of flipped nodes in the neighbor of each node is approximately equal to the fraction $f$ of edges coming from flipped nodes in a network. The dynamics of degree-correlated scale-free networks under random stimuli has the same characteristic as uncorrelated networks: the attractor systems respond to random stimuli as a whole. Under directed stimuli, the variation of $f_c$ versus the Pearson correlation coefficient indicates that the dynamics of directed stimulated attractor networks is affected seriously by degree correlation.

Next we numerically investigate the dynamical process of the evolution of the attractor system in the case of directed stimuli and reveal the underlying mechanism of the effect of degree correlation. To do this, we give a directed stimuli with size equal to 235 to a realization of the uncorrelated network. The stimulus is larger than the average attractor basin for uncorrelated scale-free attractor systems given in Fig. 1 which is equal to 215(±12). In Fig. 3 the dynamical process of the evolution of the system is represented by the number of flipped nodes ($N_f$). At the first step of the evolution, the number of the flipped nodes is 488, which is near half of the network size. And then the system evolves to another imprinted functional state, as shown in the inset of Fig. 3. The evolution shows that the uncorrelated scale-free networks response to directed stimuli as a whole, as the above analysis.

For assortative networks, we give a directed stimulus with the size 170 to attractor systems. Although the size of the stimuli is smaller than the mentioned average attractor basin of uncorrelated networks, the system responds to the stimulus with the process of the change of the system state, as shown in Fig. 4. We note that the number of flipped nodes increases gradually. In contrast with uncorrelated scale-free networks, the evolution shows that the assortative scale-free network system does not make response as a whole. In assortative scale-free networks, nodes of a large degree preferentially connect to nodes of the greatest degrees—i.e., stimulated nodes—and thus they are easier to get the condition for changing their states. So the set of flipped nodes can be extended by assortative mixing. The assortative scale-free network system evolves as a hierarchical cascade [23] that progresses from higher- to lower-degree classes. Therefore the basin of attraction of the assortative network system de-

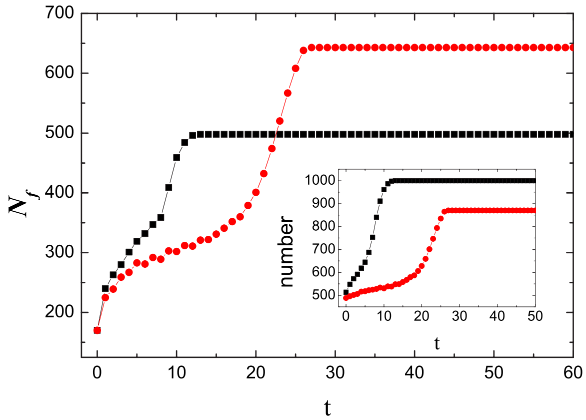


FIG. 4. (Color online) The number of flipped nodes in the process of the evolution of two assortative systems with $r=0.13$ (squares) and $r=0.15$ (circles). Inset: the number of nodes whose state $s_i(t)$ is the same as $s_i^2$.

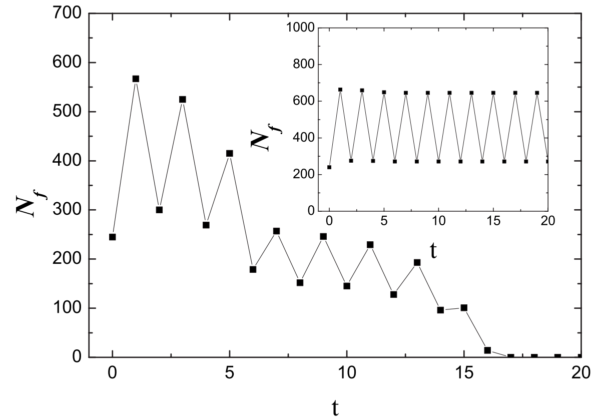


FIG. 5. The number of flipped nodes in the process of disassortative system, $r=-0.16$. Inset: system with $r=-0.30$. The size of stimuli is 245.

creases and the system is more sensitive to directed stimuli.

With an increase of the Pearson correlation coefficient, the cluster coefficient of assortative networks is increased by the degree-based reshuffling steps [5]. The cluster property also affects the dynamics of assortative scale-free networks. In Fig. 4 we show two numerical simulations with different types of dynamics. For one kind of dynamics (square), the stable system states are the functional states imprinted by the Hebbian rule, as the uncorrelated networks. The upper curve (square) in the inset of Fig. 4 shows that a system evolves into the second functional state. For another kind of dynamics (circle), the stable system state at the end of evolution is not the imprinted functional state. The lower curve (circle) in the inset of Fig. 4 shows the discrepancy. In this kind of systems, a cluster forms between stimulated nodes which have a high density of edges within them, with a lower density of edges between other groups of nodes. So these stimulated nodes hold their states on $-s_i^1$. Additionally, the state of some low-degree nodes which connect tightly to the cluster is also held. These nodes held by the cluster structure result in the difference between the system state and the imprinted functional state. There is a critical value $r_c$, for the networks used in simulations $r_c=0.32$, below which two types of dynamics are possible (and larger the value of $r$ is, the more frequently the second type of dynamics occurs), while above which systems only respond to stimuli by the second type of dynamics. Because of the cluster property of assortative networks, too large assortative mixing is not expected for the response of networks. In the limit of $r \rightarrow 1$, networks disintegrate into isolated clusters, each of them consisting of nodes with a certain degree $k$. Directed stimuli cannot induce these systems to change their functional states, but only change a few clusters and leave the other nodes in their initial states.

For the disassortative system, we choose a reshuffling scale-free network realization with Pearson correlation coefficient $r=-0.16$ which has the lowest sensitivity to directed stimuli as shown in Fig. 1. We give the disassortative network a directed stimulus with size 245 which is larger than the average attractor basin of the uncorrelated scale-free networks. Figure 5 shows the dynamical process of the evolution of the system. Although more than half of the nodes flip their states at the first step, the system state is attracted to the original functional state. In disassortative networks, nodes with large degrees preferentially connect to nodes with small ones. Under directed stimuli, the fraction of stimulated nodes in the neighbors of the nodes in the middle degree class is less than the fraction of the edge coming from stimulated nodes. Thus, more nodes need to be stimulated than uncorrelated systems for inducing the system into a random state, and the basin of attraction of disassortative system is extended.

For larger disassortative mixing systems, the second imprinted functional state cannot be reached. The inset of Fig. 5 shows the dynamical process of evolution of a network realization with $r=-0.30$. The system is induced into a stable oscillation state, which is established by the interaction between large and small nodes. The system with a large disassortative mixing property is easier to respond to the directed stimuli by evolving into stable oscillation states. This structural property leads to the nonmonotonic behavior of sensitivity versus Pearson correlation coefficient shown in Fig. 1. Additionally, it is notable that the too large disassortative degree correlation also destroys the ability of systems to respond to directed stimuli with imprinted functional states, as the too large assortative degree correlation.

In summary, we have studied the effect of the degree correlation on the response of scale-free networks to stimuli. Correlated scale-free networks retain the robustness to random stimuli. In the region of the Pearson correlation coefficient in which we are interest, assortative scale-free networks are more sensitive to directed stimuli than uncorrelated ones and the sensitivity of scale-free networks is weaken when networks are disassortative. We found that the effects of degree correlation result from the properties of the dynamics of degree-correlated network systems. Uncorrelated networks respond to stimuli as a whole, while the degree correlation of a network destroys the identical critical condition of all nodes for the response to directed stimuli. Assortative scale-free networks reduce the need for the size of directed stimuli to respond via a cascade that progresses from higher- to lower-degree classes. The disassortative correlation extends the size of the basin of attraction by the nodes in the middle degree class which has less stimulated neighbors and stays on the initial state. But the response of too large assortative and disassortative scale-free networks is destroyed by the structure property and imprinted functional states cannot be reached. Since many realistic complex networks have both scale-free and degree-correlated properties, the intuitive description of the dynamics might contribute to understanding of the attributes of realistic networks.

This work was supported by the Fundamental Research Fund for Physics and Mathematics of Lanzhou University under Grant No. Lzu05008. X.-J. X. acknowledges financial support from FCT (Portugal), Grant No. SFRH/BPD/30425/2006.

[1] Y. Bar-Yam and I. R. Epstein, Proc. Natl. Acad. Sci. U.S.A. **101**, 4341 (2004).
[2] A. Vázquez and Y. Moreno, Phys. Rev. E **67**, 015101(R) (2003).
[3] R. Pastor-Satorras, A. Vázquez, and A. Vespignani, Phys. Rev. Lett. **87**, 258701 (2001).
[4] M. E. J. Newman, Phys. Rev. Lett. **89**, 208701 (2002).
[5] R. Xulvi-Brunet and I. M. Sokolov, Phys. Rev. E **70**, 066102 (2004).
[6] M. E. J. Newman, Phys. Rev. E **67**, 026126 (2003).
[7] A. Vázquez, M. Boguñá, Y. Moreno, R. Pastor-Satorras, and A. Vespignani, Phys. Rev. E **67**, 046111 (2003).
[8] A. Capocci, G. Caldarelli, and P. De Los Rios, Phys. Rev. E **68**, 047101 (2003).
[9] M. E. J. Newman and J. Park, Phys. Rev. E **68**, 036122 (2003).
[10] J. Berg and M. Lässig, Phys. Rev. Lett. **89**, 228701 (2002).
[11] K.-I. Goh, E. Oh, B. Kahng, and D. Kim, Phys. Rev. E **67**, 017101 (2003).
[12] S. Maslov and K. Sneppen, Science **296**, 910 (2002).
[13] P. L. Krapivsky and S. Redner, Phys. Rev. E **63**, 066123 (2001).
[14] S. N. Dorogovtsev, Phys. Rev. E **69**, 027104 (2004).
[15] M. Boguñá and R. Pastor-Satorras, Phys. Rev. E **66**, 047104 (2002).
[16] L. K. Gallos and P. Argyrakis, Phys. Rev. E **72**, 017101 (2005).
[17] G. Bianconi and M. Marsili, Phys. Rev. E **73**, 066127 (2006).
[18] A. Fronczak and P. Fronczak, Phys. Rev. E **74**, 026121 (2006).
[19] J. J. Hopfield, Proc. Natl. Acad. Sci. U.S.A. **79**, 2554 (1982).
[20] D. J. Amit, H. Gutfreund, and H. Sompolinsky, Phys. Rev. Lett. **55**, 1530 (1985).
[21] A.-L. Barabási and R. Albert, Science **286**, 509 (1999).
[22] For $r \approx 0$, we have checked that the size of the attractor basin does not change when using neutral shuffled networks instead of BA networks.
[23] M. Barthélemy, A. Barrat, R. Pastor-Satorras, and A. Vespignani, Phys. Rev. Lett. **92**, 178701 (2004).